A generalization of Gaschütz's theorem of sylowizers

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Let G be a finite group and R a p-subgroup of G. The subgroup S of G is a sylowizer of R if R is a Sylow p-group in S and S is maximal with this property (in G) [1]. GASCHÜTZ has given two sufficient conditions such that all sylowizers of R in G are conjugate. ISAACS has given an example where the sylowizers of a p-group R in a finite group G are not conjugate, that is, the conjugacy of the sylowizers of R is not true in general. In the first result GASCHÜTZ proves that if G is a solvable (finite) group, P is a Sylow p-group of G and $R \lhd P$, then all sylowizers of R in G are conjugate. As it is well known if G is a finite solvable group then the sylowizer of R has the factorization RC where $(|R|, |C|)=1$, that is, C is a complement of R in the sylowizer. In this note we give a generalization of the theory of GASCHÜTZ.

1. Let G be a finite group. Π the set of all prime divisors of $|G|$, $\pi \subset \Pi$, $\pi' = \Pi \setminus \pi$. We say that G satisfies the condition $D_{\pi'}$ of HALL if G has a Hall π'-subgroup H and every π'-subgroup of G is contained in a conjugate of H.

A relation ϱ defined on the set of the subgroups of G is said to be a *T-relation* if 1) $R\varrho C$ implies that R is a π-subgroup and C is a π'-subgroup; 2) $R\varrho\{1\}$ for every π-subgroup R; 3) if $\langle C_1, C_2\rangle$ is a π'-subgroup then $R\varrho G$, $R\varrho C_2$ imply $R\varrho\langle C_1, C_2\rangle$; 4) $R\varrho C$ and $a \in G$ imply $(a^{-1}Ra)\varrho(a^{-1}Ca)$.

Furthermore, let ϱ be a T-relation defined in G, and R a π-subgroup in G. A subgroup C of G is an *absolute ϱ-complement* of R if: 1) $R\varrho C$ holds; 2) C is maximal with respect to 1), that is, there does not exist a π'-subgroup such that $C \subset D$ and $R\varrho D$.

Let H be a Hall π'-subgroup of G. The subgroup C of G is a *relative ϱ-complement* of R in H if: 1) $C \subseteq H$; 2) $R\varrho C$ holds; 3) C is maximal with respect to 1) and 2).

Remark 1. If R is a π-subgroup of G, C is an absolute ϱ-complement of R, H is a Hall π'-subgroup of G and $C \subseteq H$, then C is at the same time a relative ϱ-complement of R.

Remark 2. If R is a π-subgroup and H is a Hall π'-subgroup of G then there exists exactly one relative ϱ-complement of R in H.

To show this, consider the set Σ of subgroups X of H for which $R\varrho X$. Then for $X_1, X_2 \in \Sigma$ we have (by the definition of the T-relations) $\langle X_1, X_2\rangle \in \Sigma$ and so there exists only one relative ϱ-complement of R in H.

Examples for T-relations:

$R\varrho_1 C$ if C is permutable with R,
$R\varrho_2 C$ if C is permutable with all elements of R,
$R\varrho_3 C$ if all elements of C are permutable with R,
$R\varrho_4 C$ if all elements of C are permutable with all elements of R.

Theorem 3. *Let G be a finite group with property $D_{\pi'}$, P a Hall π-subgroup of G and R a subgroup of P. Let ϱ be a T-relation between the π-subgroups and π'-subgroups of G and C an absolute ϱ-complement of R of maximal order. The relative ϱ-complements of R (in various Hall π'-subgroups of G) are all conjugates of C with respect to P and so are all absolute ϱ-complements of C if and only if $a^{-1}Ra\varrho C$ for every $a\in P$.*

Proof. It is easy to see that the condition is necessary. Indeed, consider an element $a\in P$. By hypothesis it is true that $R\varrho aCa^{-1}$ whence $a^{-1}Ra\varrho C$.

The condition is sufficient. By hypothesis $a^{-1}Ra\varrho C$ for every $a\in P$. First of all we prove that aCa^{-1} is a relative ϱ-complement for every $a\in P$ (in a convenient Hall π'-subgroup of G). Because $a^{-1}Ra\varrho C$ and ϱ is a T-relation, $R\varrho aCa^{-1}$ holds. Since $D_{\pi'}$ is true, there exists a Hall π'-subgroup H in G which contains the subgroup C, that is, $aCa^{-1}\subseteqq aHa^{-1}$ where aHa^{-1} is a Hall π'-subgroup of G. By Remark 2 there is only one relative ϱ-complement $\bar{C}$ of R in aHa^{-1}. The subgroup $\bar{C}$ is contained in an absolute ϱ-complement $\bar{\bar{C}}$ of R. Thus we get $aCa^{-1}\subseteqq\bar{C}\subseteqq\bar{\bar{C}}$ and $C\subseteqq a^{-1}\bar{C}a\subseteqq a^{-1}\bar{\bar{C}}a$. But C is an absolute ϱ-complement of R from which $C=a^{-1}\bar{\bar{C}}a$ follows. So we have $aCa^{-1}=\bar{\bar{C}}$. The subgroup aCa^{-1} is a conjugate of C with respect to P and so every conjugate of C with respect to P is an absolute ϱ-complement of R. Finally (by Remark 1) aCa^{-1} is a relative ϱ-complement in every Hall π'-subgroup of G which contains it.

It remains to prove that, conversely, if $\bar{C}$ is a relative ϱ-complement of R contained in a Hall π'-subgroup $\bar{H}$ of G then $\bar{C}$ is a conjugate of C with respect to P. $D_{\pi'}$ holds in G, so we have $C\subseteqq d^{-1}\bar{H}d$ $(d\in G)$. The subgroup $H=d^{-1}\bar{H}d$ is a Hall π'-subgroup of G and we have the factorisation $G=PH$ $(P\cap H=1)$. Hence $d=ab$ $(a\in P, b\in H)$ from which $\bar{H}=dHd^{-1}=abHb^{-1}a^{-1}=aHa^{-1}$, that is, $H=a^{-1}\bar{H}a$ $(a\in P)$. Since $C\subseteqq H=a^{-1}\bar{H}a$ one gets $aCa^{-1}\subseteqq\bar{H}$. Because $a\in P$ and by hypothesis $a^{-1}Ra\varrho C$ therefore (ϱ being a T-relation) $R\varrho aCa^{-1}$. Because $aCa^{-1}\subseteqq\bar{H}$ and since $\bar{C}$ is the only relative ϱ-complement of R in $\bar{H}$ (by Remark 2) we get $aCa^{-1}\subseteqq\bar{C}$. The subgroup $\bar{C}$ is contained in an absolute ϱ-complement $\bar{\bar{C}}$ of R and therefore $C\subseteqq a^{-1}\bar{C}a\subseteqq a^{-1}\bar{\bar{C}}a$.

But then $C=a^{-1}\bar{\bar{C}}a$ because C is supposed to be an absolute ϱ-complement of R. So we get $aCa^{-1}=\bar{C}=\bar{\bar{C}}$, and Theorem 3 is proved.

Corollary 4. *Let G be a finite group with property $D_{\pi'}$, P a Hall π-subgroup of G and R a normal subgroup of P. Let ϱ be a T-relation between π- and π'-subgroups. Then the relative ϱ-complements of R (in the various Hall π'-subgroups of G) are conjugate with respect to P (and so are absolute ϱ-complements, too).*

Proof. Let C be an absolute ϱ-complement of R. Then $R\varrho C$ and $a^{-1}Ra\varrho C$ for every $a\in P$ ($a^{-1}Ra=R$). By Theorem 3 all relative ϱ-complements of R (in the various Hall π'-subgroups of G) are conjugates of C with respect to P and thus they are absolute ϱ-complements of R, too.

2. Consider now the T-relation ϱ_1. Let G be a finite group and R a π-subgroup of G. We say that the subgroup S of G is a *π-sylowizer* of R if 1) R is a Hall π-subgroup of S, 2) S is maximal with respect to 1). If π contains only one prime number p then the π-sylowizer coincides with the sylowizer concept of Gaschütz [1].

Theorem 5. *Let G be a finite group such that every subgroup of G has the property $D_{\pi'}$. Let R be a π-subgroup of G. Then S is a π-sylowizer of R if and only if $S=RC$ where C is an absolute ϱ_1-complement of R.*

Proof. Let C be an absolute ϱ_1-complement of R, that is, C and R are permutable and RC is a subgroup of G. R is a π-subgroup, C is a π'-subgroup which means that R is a Hall π-subgroup in RC. We prove now that RC is a sylowizer of R. Otherwise there would be a subgroup S of G such that $RC\subset S$ and R is a Hall π-subgroup of S. But by hypothesis S satisfies $D_{\pi'}$, that is, C is contained in a Hall π'-subgroup $\bar{C}$ in S with $S=R\bar{C}$. Hence $R\varrho_1\bar{C}$, and because of $C\neq\bar{C}$ we have a contradiction since C is an absolute ϱ_1-complement of R.

Conversely, suppose that S is a π-sylowizer of R. Since every subgroup has property $D_{\pi'}$, so does a Hall π'-subgroup C. By hypothesis R is a Hall π-subgroup of S from which we have $S=RC$, that is, $R\varrho_1 C$. We prove that C is an absolute ϱ_1-complement of R. Otherwise C would be contained in an absolute ϱ_1-complement $\bar{C}$ of R. Because of $R\varrho_1\bar{C}$ the subgroup $\bar{C}$ is permutable with R and in the same time $S=RC\subset R\bar{C}$ where R is a Hall π-subgroup of $R\bar{C}$. This is a contradiction because by hypothesis S is a sylowizer of R. It follows that C is an absolute ϱ_1-complement of R.

Theorem 6. *Let G be a finite group such that every subgroup of G has the property $D_{\pi'}$. Let P be a Hall π-subgroup of P. Then the π-sylowizers of R are conjugate with respect to P.*

Proof. Let S_1 and S_2 be two π-sylowizers of R. By Theorem 5 we have $S_1=$ $=RC_1$, $S_2=RC_2$ where C_1 and C_2 are absolute ϱ_1-complements of R. By Corollary 4 there exists an element $a \in P$ such that $a^{-1}C_1a=C_2$. It follows $a^{-1}S_1a=a^{-1}RC_1a=$ $=(a^{-1}Ra)(a^{-1}C_1a)=RC_2=S_2$, that is, S_1 and S_2 are conjugate with respect to P.

In particular, the conditions of Theorem 6 are satisfied for solvable groups. If π contains only one prime number then we get Theorem 1 of GASCHÜTZ [1].

Theorem 7. *Let G be a finite group such that every subgroup of G has the property $D_{\pi'}$. Let P be a Hall π-subgroup of G and R a subgroup of P. Suppose that $G=PH$ where H is a Hall π'-subgroup of G. Then the π-sylowizers of R are conjugate if one of the following conditions is satisfied:*

a) $G=N_G(R)\cdot N_G(H)$,

b) $G=N_G(C_1)\cdot N_G(H)$,

c) $G=N_G(R)\cdot N_G(C_1)\cdot N_G(H)$,

where C_1 is a relative ϱ_1-complement of R in a sylowizer of R.

Proof. a) Let S_1 and S_2 be two sylowizers of R. By Theorem 5 we have $S_1=RC_1$, $S_2=RC_2$. We may suppose that $C_1 \subseteqq H$ (property $D_{\pi'}$). Again by property $D_{\pi'}$ the subgroup C_2 is contained in a conjugate $a^{-1}Ha$ where $a \in N_G(R)$ in virtue of a). However then C_1 and C_2 as the only relative ϱ_1-complements in H and in $a^{-1}Ha$, respectively, are also conjugate: $C_2=a^{-1}C_1a$, that is, $S_2=RC_2=$ $=Ra^{-1}C_1a=a^{-1}Ra\,a^{-1}C_1a=a^{-1}S_1a$.

b) In this case we prove that $S_1=RC_1$ is the only sylowizer of R in G. Suppose that R has two sylowizers $S_1=RC_1$, $S_2=RC_2$ $(C_1 \subseteqq H)$. By hypothesis C_2 is contained in $b^{-1}Hb$ where $b \in N_G(C_1)$. It follows $C_1 \subseteqq H \cap b^{-1}Hb$. But RC_1 and RC_2 are sylowizers of R and C_1, C_2 are contained in $b^{-1}Hb$. So we have $C_1=C_2$.

c) Again let $S_1(=RC_1)$ and $S_2(=RC_2)$ be two sylowizers of R in G and $C_1 \subseteqq H$. By hypothesis (property $D_{\pi'}$) there are two elements $a \in N_G(R)$, $b \in N_G(C_1)$ such that $C_2 \subseteqq abHb^{-1}a^{-1}$ from which we get $C_2=ab\;C_3b^{-1}a^{-1}$ with $C_3 \subseteqq H$. Hence $a^{-1}Ra=R$ is permutable with bC_3b^{-1}. But $bC_3b^{-1} \subseteqq bHb^{-1}$, that is, C_1 and bC_3b^{-1} are relative ϱ_1-complements of R in bHb^{-1}, that is, $C_1=C_3$. Thereby we get $C_2=$ aC_1a^{-1}.

Reference

[1] W. GASCHÜTZ, Sylowizatoren, *Math. Z.*, **122** (1971), 319—320.

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